1. Redo question 6 of Homework 4 to practice juggling with ordinals.

2. Let $(A,<)$ be a total ordering, α an ordinal, and $f\colon (A,<) \to (\alpha,\in)$ an order-preserving function.

(a) Prove that $(A,<)$ is a well-ordering and $\mathtt{tp}(A,<) \leqslant \alpha$.

(b) Conclude that if $\pi\colon \beta \to \beta$ is an order-preserving function, then $\sup \pi''\beta = \beta$.

(c) Give an example of $(A,<)$, $\alpha \neq \omega$, and non-surjective f such that $\mathtt{tp}(A,<) = \alpha$.

3. Prove without using Axiom of Choice that if a set A is Dedekind infinite, then $\omega \sqsubseteq A$.

4. For a set A, write down all of the statements you know that are equivalent to "A is *finite*" and prove these equivalences. Do the same for *countable*.

5. Prove that for any infinite set A and a finite subset $F \subseteq A$, $A \equiv A \setminus F$.

6. Show that for a countable set A, the set $\mathscr{P}_{<\omega}(A)$ of all finite subsets of A is countable.

7. Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be defined by $f(n) := 2n$ and $g(n) := 3n$. Explicitly define a bijection $h\colon \mathbb{N} \to \mathbb{N}$ such that $h \subseteq f \cup g^{-1}$, where by g^{-1} we mean the graph of g with the swapped coordinates, i.e., $g^{-1} := \{(y,x) \in \mathbb{N}^2 : g(x) = y\}$.

8. Let E_0 denote the relation of *eventual equality* on $2^{\mathbb{N}}$, i.e., for $x, y \in 2^{\mathbb{N}}$,

$$x\, E_0\, y :\Longleftrightarrow \exists m \in \mathbb{N}\ \forall n \geqslant m\ [x(n) = y(n)].$$

(a) Prove that E_0 is an equivalence relation.

(b) **The prisoners and hats problem.** ω-many prisoners are sentenced to death, but they can get out under the following condition. On the day of the execution they will be lined up, i.e., enumerated $(p_n)_{n\in\mathbb{N}}$, so that everybody can see everybody else but themselves. Each of the prisoners will have a red or blue hat put on them, but he/she won't be told which color it is (although they can see the other prisoners' hats). On command, all the prisoners at once make a guess as to what color they think their hat is. If all but finitely many prisoners guess correctly, they all go home free; otherwise all of them are executed. The good news is that the prisoners think of a plan the day before the execution, and indeed, all but finitely many prisoners guess correctly the next day, so everyone is saved. How do they do it?

9. Do Homework 7, especially 1, 2(a)–(c), 3, and 4.

10. Prove that any vector space V admits a basis.

HINT: A set $A \subseteq V$ is a basis if and only if it is an $\subseteq$-maximal linearly independent set.